# MATH 105A and 110A Review: Elementary matrices and row operations

## Facts to Know:

Elementary row operations on a matrix $A$:

1. **(Row switching)**

$$\begin{bmatrix} - R_j - \\ - R_i - \end{bmatrix} \quad R_j \longleftrightarrow R_i \quad \begin{bmatrix} - R_i - \\ - R_j - \end{bmatrix}$$

2. **(Row addition)**

$$\begin{bmatrix} - R_i - \\ - R_j - \end{bmatrix} \quad R_i + cR_j \longrightarrow R_i \quad \begin{bmatrix} - R_i + cR_j - \\ - R_j - \end{bmatrix}$$

3. **(Row multiplication)** $c \neq 0$

$$\begin{bmatrix} - R_i - \end{bmatrix} \quad c \cdot R_i \longrightarrow R_i \quad \begin{bmatrix} - c \cdot R_i - \end{bmatrix}$$

The matrix $A$ is in **echelon form** if the following three properties hold:

1. Rows of zeros are below any non-zero row.
2. Any leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zero.

Picture of echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

- The $\blacksquare$'s correspond to non-zero numbers and are called pivots.
- The columns with $\blacksquare$'s are called pivot columns.

If the matrix $A$ is in echelon form and additionally satisfies the following two conditions, then $A$ is in **reduced echelon form**:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

## Examples:

1. Identify which matrices are in echelon form, reduced echelon form, or neither.

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

e.f.
r.e.f.

$$\begin{bmatrix} \boxed{1} & \cancel{1} & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

e.f.
Not in r.e.f

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Not in e.f.
not in r.e.f.

$$\begin{bmatrix} \boxed{2} & 0 & 0 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & \boxed{4} \\ 0 & 0 & 0 & \cancel{1} \end{bmatrix}$$

not in e.f.
not in r.e.f.

2. Reduce $A$ to reduced echelon form.

$$A = \begin{bmatrix} \boxed{1} & 3 & 1 & 3 & 3 \\ 0 & \boxed{2} & 0 & 2 & 2 \\ 0 & 0 & 0 & \boxed{3} & 3 \end{bmatrix}.$$

$$A \quad \begin{matrix} \tfrac{1}{2}R_2 \to R_2 \\ \tfrac{1}{3}R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 - 3R_2 \to R_1 \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 - R_1 \to R_2 \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

## Facts to Know:

An **elementary matrix** $E$ is obtained by applying one of the row operations to the identity matrix $I$.

Each elementary row operation can be carried out by multiplying A from the left by the corresponding elementary matrix.

## Examples:

3. Carry out row operation $R_1 + 2R_2 \to R_1$ on matrix $A$ using an elementary matrix, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1/2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2 \to R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Check our work:

$$\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_1 + 2R_2 \to R_1} \begin{bmatrix} 3 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$